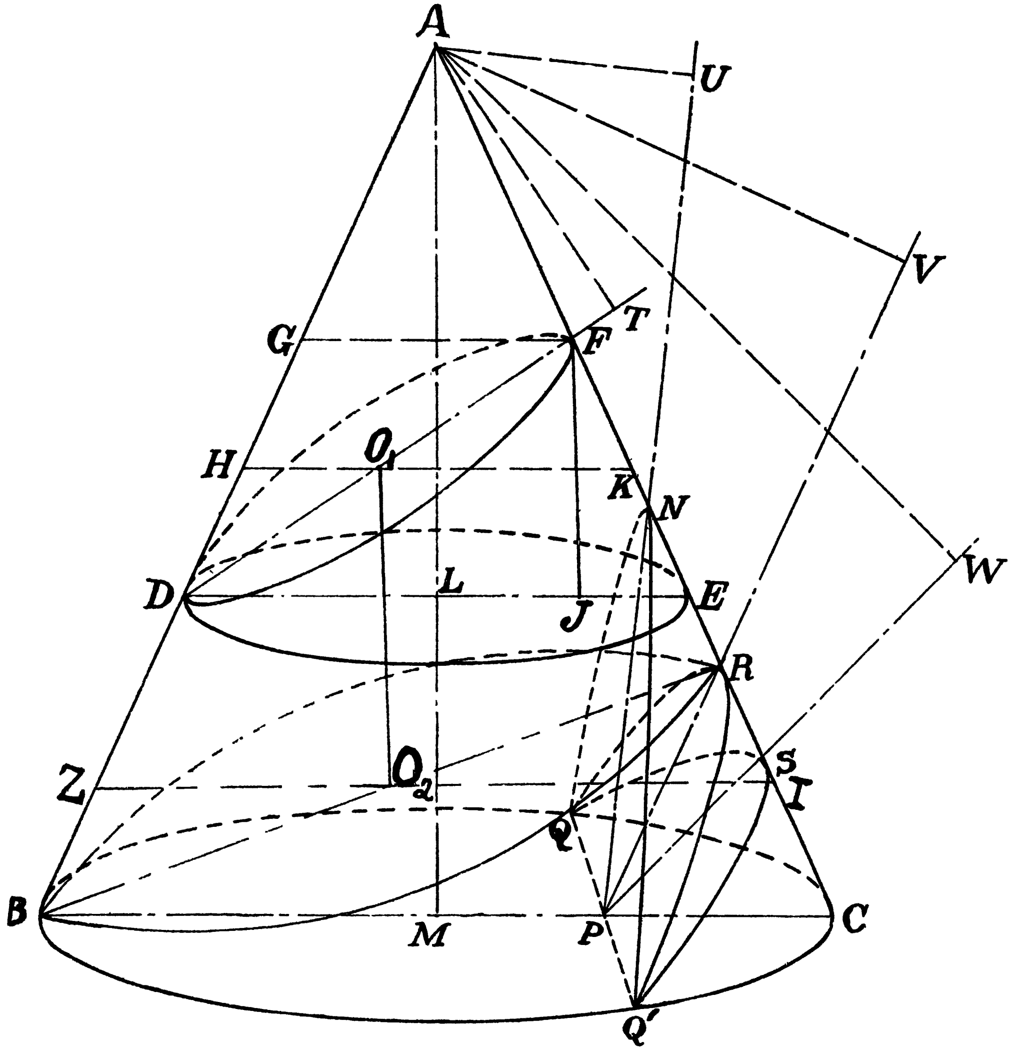
MECHANICS.

196. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From a uniform, solid right circular cone two planes cut off a portion such that the sections are similar ellipses with co-planar axes (not parallel). The centers of the elliptic faces are O_1, O_2, and the center of gravity of the solid is G. GX parallel to O_1O_2 cuts the axis of the cone in X. Find GX/O_1O_2 in terms of the ratio of the major axes of the ellipses.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the cone; BR, DF the major axes of the required ellipses. For calculations we will only consider the cone ADE and the plane DF.



Let $x^2+z^2=n^2(c-y)^2$ be the cone, $x=my-R$ the plane, where $R=DL$, $c=AL$, $h=FJ$, $r=\frac{1}{2}GF$, $n=\tan LAE=(R-r)/h$, $m=\tan FDE=(R+r)/h$, $c=Rh/(R-r)$. Let $my-R=x_1$, $n(c-y)=x_2$, $(x+nc)/n=y_1$, $(x+R)/m=y_2$, $(nc-x)/n=y_3$.

$$\therefore v=2\int_0^h\int_{-x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}\,dydx+2\int_h^c\int_{-x_2}^{x_2}\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=\int_0^h\left\{n^2(c-y)^2\left[\tfrac{1}{2}\pi+\sin^{-1}\left(\frac{my-R}{n(c-y)}\right)\right]\right.$$

$$\left.+(my-R)\sqrt{[n^2(c-y)^2-(my-R)^2}\right\}dy+\pi n^2\int_h^c(c-y)^2dy=\frac{\frac{1}{3}\pi Rrh\sqrt{Rr}}{R-r}.$$

$$\therefore v=\tfrac{1}{3}\pi\,(AD.AF)^{\frac{3}{2}}\sin^2\beta\cos\beta,\text{ where }\angle LAE=\beta.$$

$$\bar{x}v=2\int_0^h\int_{-x_2}^{x_1}x\sqrt{]n^2(c-y)^2-x^2]}\,dydx+2\int_h^c\int_{-x_2}^{x_2}x\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=-\tfrac{2}{3}\int_0^h[n^2(c-y)^2-(my-R)^2]^{\frac{3}{2}}\,dy$$

$$=-\frac{2}{3\,(m^2-n^2)^{\frac{5}{2}}}\int_{-u_1}^{u_1}[(ncm-Rn)^2-u^2]^{\frac{3}{2}}\,du$$

where $u=(m^2-n^2)y+n^2c-Rm$, $u_1=2rR/h$.

$$\therefore\bar{x}v=-\tfrac{1}{8}\pi Rrh\sqrt{(Rr)}=-\tfrac{1}{8}\pi(AD-AF)\,(AD.AF)^{\frac{3}{2}}\sin^3\beta\cos\beta.$$

$$\bar{y}v=2\int_0^h\int_{-x_2}^{x_1}y\sqrt{[n^2(c-y)^2-x^2]}\,dydx+2\int_h^c\int_{-x_2}^{x_2}y\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=2c\int_0^h\int_{-x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}\,dydx+2c\int_h^c\int_{-x_2}^{x_2}\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$-2\int_{-r}^{r}\int_{y_2}^{h}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dxdy$$

$$-2\int_{-R}^{-r}\int_{y_2}^{y_1}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dxdy$$

$$-2\int_{-r}^{r}\int_{h}^{y_3}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dx\,dy$$

$$=\frac{\frac{1}{3}\pi cRrh\sqrt{Rr}}{R-r}-\frac{2}{3n^2m^3}\int_{-R}^{r}[n^2(cm-R-x)^2-m^2x^2]^{\frac{3}{2}}\,dx=\frac{\frac{1}{3}\pi cRrh\sqrt{Rr}}{R-r}$$

$$\frac{2}{3n^2m^3(m^2-n^2)^{\frac{5}{2}}}\int_{-u_2}^{u_2}[(ncm^2-nRm)^2-x^2]^{\frac{3}{2}}\,du,$$

where $u=(m^2-n^2)x^2+cmn^2-Rn^2$, $u_2=\dfrac{2Rr(R+r)}{h^2}$.

$$\therefore \bar{y}v=\frac{\pi Rrh^2\sqrt{(Rr)}\,(5R-3r)}{24(R-r)^2}=\frac{\pi}{24}(5AD-3AF)\,(AD.AF)^{\frac{3}{2}}\sin^2\beta\cos^2\beta.$$

$\bar{x}$, $\bar{y}$ give the center of gravity of the cone ADF. The center of gravity G of the portion $FDBRF$ is given as follows:

$$\bar{x}_1=\frac{3[(AD-AF)(AD.AF)^{\frac{3}{2}}-(AB-AR)(AB.AR)^{\frac{3}{2}}]\sin\beta}{8[(AB.AR)^{\frac{3}{2}}-(AD.AF)^{\frac{3}{2}}]}=a.$$

$$\bar{y}_1=\frac{[(5AB-3AR)(AB.AR)^{\frac{3}{2}}-(5AD-3AF)(AD.AF)^{\frac{3}{2}}.]\cos\beta}{8[(AB.AR)^{\frac{3}{2}}-(AD.AF)^{\frac{3}{2}}]}=b.$$

Now $DF=\sqrt{(AD^2+AF^2-2AD.AF\cos2\beta)}$,
$BR=\sqrt{(AB^2+AR^2-2AB.AR\cos2\beta)}$.
Coordinates of O_1 are, $[AB-\frac{1}{2}(AD+AF)]\cos\beta$, and, $\frac{1}{2}(AD-AF)\sin\beta$,
Coordinates of O_2 are, $\frac{1}{2}(AB-AR)\cos\beta$, and, $\frac{1}{2}(AB-AR)\sin\beta$.
$\therefore O_1O_2=\frac{1}{2}](AB-AD)^2+(AR-AF)^2+2(AB-AD)(AR-AF)\cos2\beta]^{\frac{1}{2}}$.

GX makes with the axis of abscissas an angle whose tangent is

$$\left(\frac{AD+AF-AB-AR}{AB+AF-AR-AD}\right)\cot\beta=\mu.$$

$\therefore y-b=\mu(x-a)$ is the equation to GX. This meets the axis of ordinates in $x=0$, $y=b-\mu a$.

$\therefore GX=a\sqrt{(1+\mu^2)}$

$$=\frac{a\sqrt{[(AB-AD)^2+(AR-AF)+2(AB-AD)(AR-AF)\cos2\beta}}{(AB+AF-AR-AD)\sin\beta}$$

$$=\frac{2aO_1O_2}{(AB+AF-AR-AD)\sin\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{2a}{(AB+AF-AR-AD)\sin\beta}$$

$$=\frac{6[(AD-AF)(AD.AF)^{\frac{3}{2}}-(AB-AR)(AB.AR)^{\frac{3}{2}}]}{8(AB+AF-AR-AD)[(AB.AR)^{\frac{3}{2}}-(AD.AF)^{\frac{3}{2}}]}.$$

Let $\angle FDE=\theta$, $\angle RBC=\phi$, semi-minor axis of ellipse $DF=p$, of ellipse $BR=q$. Then

$$p=\sqrt{(HO_1.O_1K)}=\sqrt{(AD.AF)}\sin\beta, \qquad q=\sqrt{(ZO_2.O_2I)}=\sqrt{(AB.AR)}\sin\beta,$$

$$AD-AF=FE=\frac{DF\sin\theta}{\cos\beta}, \qquad AB-AR=RC=\frac{BR\sin\phi}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6(p^3DF\sin\theta-q^3BR\sin\phi)}{8(BR\sin\phi-DF\sin\theta)(q^3-p^3)}.$$

Let $DF=2P$, $BR=2Q$. Then

$$p=\frac{P\sqrt{[\cos(\theta+\beta)\cos(\theta-\beta)]}}{\cos\beta}=\frac{P\sqrt{[\cos^2\theta-\sin^2\beta]}}{\cos\beta},\quad q=\frac{Q\sqrt{[\cos^2\theta-\sin^2\beta]}}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6[P^4\sin\theta(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}-Q^4\sin\phi(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}]}{8\{[Q\sin\phi-P\sin\theta][Q^3(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}-P^3(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}]\}}.$$

If the planes of the ellipses are parallel, $\theta=\phi$, and we get

$$\frac{GX}{O_1O_2}=\frac{6(P^4-Q^4)}{8(Q-P)(Q^3-P^3)}=\frac{6(P^2+Q^2)(P+Q)}{P^3-Q^3}.$$

MISCELLANEOUS.

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165. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $\tan^{-1}\dfrac{n}{n+1}+\tan^{-1}\dfrac{1}{2n+1}=\frac{1}{4}\pi$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Me.; J. EDWARD SANDERS, Reinersville, O.; FRANK M. DRYZER, A. B., Knoxville, Tenn.; and PROF. J. W. NICHOLSON, State University, La.

Let $\alpha=\tan^{-1}\dfrac{n}{n+1}$ and $\beta=\tan^{-1}\dfrac{1}{2n+1}$.

Then $\tan\alpha=\dfrac{n}{n+1}$, $\tan\beta=\dfrac{1}{2n+1}$, and $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\dfrac{n}{n+1}+\dfrac{1}{2n+1}}{1-\dfrac{n}{(n+1)(2n+1)}}=\frac{2n^2+2n+1}{2n^2+2n+1}=1.$$

$\therefore \alpha+\beta=\frac{1}{4}\pi$.

Also solved by G. W. Greenwood.

166. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.